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Partial Stability in Probability of Nonlinear Stochastic Discrete-Time Systems with Delay

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Abstract—A system of nonlinear stochastic functional-difference equations with finite delay is considered. By assumption, this system has a "partial" trivial equilibrium (with respect to part of the state variables). The problem under study is to analyze partial stability in probability of this equilibrium: stability is considered with respect to part of the variables determining it. The problem is solved using a discrete-stochastic modification of the method of Lyapunov–Krasovskii functionals. Conditions for partial stability in probability are established. An example is provided to illustrate the features of the proposed approach and the rationale for introducing a one-parameter family of functionals.

Keywords: system of nonlinear stochastic functional-difference equations with finite delay, partial stability in probability, method of Lyapunov–Krasovskii functionals

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1. INTRODUCTION

In the qualitative analysis and design of nonlinear dynamic systems, a separate area of research is associated with investigating the stability of systems of stochastic discrete (finite-difference) equations [1–9]. The interest in such systems is caused by the implementation of digital control systems, modeling problems in various fields, and numerical solution problems for systems of stochastic differential equations.

Within this research area, systems of discrete equations of order $m \geqslant 1$,

$$x(k+1) = X(k, x(k), x(k-1), \dots, x(k-m)),$$

are interpreted as systems of *discrete equations with finite delay*; for example, see the monographs [9–11]. This interpretation provides new capabilities for the qualitative analysis of such systems, although they can be transformed into standard one-step systems of discrete equations by introducing new variables and extending the state space.

On the other hand, in networked control applications, there arise systems of discrete equations with *variable* delay:

$$x(k+1) = X(k, x(k), x(k-\tau_1(k)), \dots, x(k-\tau_l(k))),$$

where functions $\tau_i(k)$ take an integer value from an interval $0 < \tau_i(k) \leqslant m$ at each discrete time instant k. When fixing the delay value, these systems can be treated as one-step discrete switched systems in the extended state space [12].

The general class of systems of nonlinear discrete equations with finite delay is defined by a system of *functional-difference* equations [9, 11, 13–19] of the form

$$x(k+1) = X(k, x_k),$$

whose state at each discrete time instant $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ is described by a discrete vector function $x_k = x(k+j)$, $j \in \mathbb{Z}_0 = \{-m, -m+1, \ldots, 0\}$ with the delayed argument; the number $m \geqslant 1$ specifies the delay value. This class of systems has been studied separately since the early 1990s and includes systems of discrete equations with constant, variable, and interval-type (single or multiple) delays.

With the transition to the functional-difference treatment, the stability of systems of nonlinear stochastic discrete equations with delay can be analyzed using a stochastic modification of the method of Lyapunov–Krasovskii functionals [9, 20–25] in the space of discrete (grid) functions. For deterministic nonlinear systems of discrete equations with delay, this approach was presented in [13–19]. The results obtained therein relate to the stability problem of the trivial equilibrium *with respect to all variables*. More general problems of *partial* stability have been actively considered in recent time. Note that their analysis significantly differs, see the review [26]. However, partial stability problems have not been studied for systems of nonlinear stochastic discrete equations with delay.

This paper is devoted to a general class of systems of nonlinear stochastic discrete equations with finite delay. By assumption, the system under consideration has a "partial" trivial equilibrium (with respect to some part of the state variables). We formulate the problem of stability in probability of this equilibrium with respect to part of the variables determining it. This problem is solved using the method of functionals representing discrete analogs of Lyapunov–Krasovskii functionals. They are widespread in the analysis of systems of nonlinear functional-differential equations with aftereffect (delay).

2. PROBLEM STATEMENT

We consider the linear finite-dimensional space $\mathbb{R}^n$ of vectors x with the Euclidean norm $|x|$ and divide the vector x into two parts: $x = (y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$. (The symbol T indicates the transpose.) The following class of nonlinear stochastic discrete (finite-difference) equations of the first order has been sufficiently investigated by now [1, 2]:

$$x(k+1) = X(k, x(k), \xi(k)),$$

where $k \in \mathbb{Z}_+$ denotes the discrete time; $x(k)$ is a sequence of state vector values; $\xi(k)$ is a sequence of independent random vectors defined on a probability space (Ω, F, P) with identical distribution laws for each $k \in \mathbb{Z}_+$. Here Ω is a space of elementary events $\{\omega\}$ with measurable sets with respect to a filtration F_k defined by a σ-algebra F and a given probability measure $P : F \to [0, 1]$.

Applications often lead to more general systems of stochastic nonlinear discrete equations with finite delay [9, 20–24, 27, 28] described by

$$x(k+1) = X(k, x_k, \xi(k)). \tag{1}$$

At each discrete time instant $k \in \mathbb{Z}_+$, their state is defined by a discrete vector function $x_k = x(k+j)$, $j \in \mathbb{Z}$, with the delayed argument.

For each $k \in \mathbb{Z}_+$, let the operator $X(k, \psi, \xi)$ defining the right-hand side of system (1) in the space $\{\psi\}$ of discrete (grid) functions $\psi(\theta)$, $\theta \in \mathbb{Z}_0$ with the norm $\|\psi\| = \max\{|\psi(0)|, |\psi(-1)|, \ldots, |\psi(-m)|\}$ be *continuous* in ψ, ξ in the domain $||\psi|| < \infty$. The initial state x_0 of system (1) is given by a set of values $x_0 = \{x(k_0), x(k_0-1), \ldots, x(k_0-m)\}$, which form a matrix of dimensions $n \times (m+1)$. Assume that this state is deterministic. Then for all $k_0 \geqslant 0, x_0$, there exist a unique random discrete process $x(k_0, x_0)$ adapted to the flow of σ-algebras F_k that is the solution of system (1) and a corresponding set of the sampled trajectories of system (1). We denote by $x(k) = x(k; k_0, x_0)$ the values of the random vector function $x(k_0, x_0)$ at step k of the process.

Due to the partition $x = (y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$ and the corresponding partition $x_k = (y_k^{\mathrm{T}}, z_k^{\mathrm{T}})^{\mathrm{T}}$, the system under consideration can be written as the two groups of equations

$$y(k+1) = Y(k, y_k, z_k, \xi(k)), \quad z(k+1) = Z(k, y_k, z_k, \xi(k)).$$

Under the condition $Y(k, 0, z_k, \xi(k)) \equiv 0$, the set $M\{x_k : y_k = 0\}$ is a "partial" equilibrium of system (1), i.e., an invariant set of this system. The existence of a "full" equilibrium $x_k = 0$ of system (1) is not required: this assumption may even contradict the common sense of the problems being solved.

The stability of the "partial" equilibrium $y_k = 0$ is considered with respect to a given part of the variables determining it (not all of them). For this purpose, we suppose that $y = (y_1^{\mathrm{T}}, y_2^{\mathrm{T}})^{\mathrm{T}}$, where the vector y_1 includes only those components of the vector y with respect to which stability will be analyzed. To expand the circle of concepts for the y_1-stability of the "partial" equilibrium $y_k = 0$, we introduce an arbitrary partition $z = (z_1^{\mathrm{T}}, z_2^{\mathrm{T}})^{\mathrm{T}}$ of the vector z into two groups of variables.

Let D_δ denote the domain of values x_0 such that $||y_0|| < \delta$, $||z_{10}|| < L$, and $||z_{20}|| < \infty$. Here the norms are defined by

$$||y_0|| = \max |y(k_0 + j)|, \quad ||z_{i0}|| = \max |z_i(k_0 + j)| \quad (i = 1, 2) \quad \text{for} \quad j \in \mathbb{Z}_0.$$

Definition 1. The "partial" equilibrium $y_k = 0$ of system (1) is said to be:

1) y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20} if, for each $k_0 \in \mathbb{Z}_+$, any arbitrarily small numbers $\varepsilon > 0$ and $\gamma > 0$, and any given number $L > 0$, there exists a number $\delta(\varepsilon, \gamma, L, k_0) > 0$ such that

$$P\left\{\sup_{k \geqslant k_0} |y_1(k; k_0, x_0)| > \varepsilon\right\} < \gamma \quad \text{for all } k \geqslant k_0 \text{ and } x_0 \in D_\delta; \tag{2}$$

2) uniformly y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20} if $\delta = \delta(\varepsilon, \gamma, L)$.

We formulate the following problem: find conditions under which the "partial" equilibrium $y(k) = 0$ of system (1) is y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20}, using the method of discrete Lyapunov–Krasovskii functionals.

This problem can also be treated as auxiliary in the stability analysis of the "partial" equilibrium $y_k = 0$ of system (1) with respect to all variables; when adding the control inputs $u = u(k, x_k)$ to system (1), a corresponding partial stabilization problem arises naturally.

Remark 1. If x_0 is a random variable with values in $\mathbb{R}^{n \times (m+1)}$ (independent of $\xi(k)$) and the inclusion $x_0 \in D_\delta$ holds almost surely, we arrive at the definitions similar to those of partial stability (see [29]).

Remark 2. These notions are very close to the following ones of partial stability: with respect to all [30] and some [31] variables of the "partial" equilibrium of systems of stochastic Itô differential equations and systems of stochastic discrete equations [8]. Also, they are very close to stability with respect to part of the variables of systems of functional-differential equations with aftereffect (delay) [32, 33].

3. CONDITIONS OF PARTIAL STABILITY IN PROBABILITY

In the space $\{\psi\}$ of discrete (grid) functions $\psi(\theta)$, $\theta \in \mathbb{Z}_0$, we consider single-valued scalar functionals $V = V(k, \psi)$, $V(k, 0) \equiv 0$, that are continuous in ψ for each $k \in \mathbb{Z}_+$ and are defined in the domain

$$k \geqslant 0, \quad ||\psi_{y1}|| < h, \quad ||\psi_{y2}|| + ||\psi_z|| < \infty. \tag{3}$$

The partition $\psi = (\psi_{y1}^{\mathrm{T}}, \psi_{y2}^{\mathrm{T}}, \psi_z^{\mathrm{T}})^{\mathrm{T}}$ corresponds to the above partition $x = (y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$ of the state vector x; $||\psi_{yi}|| = \max |\psi_{yi}(\theta)|$ $(i = 1, 2)$, $||\psi_z|| = \max |\psi_z(\theta)|$ for $\theta \in \mathbb{Z}_0$.

The vector function $x_k = x_k(k_0, x_0)$ defines a discrete element of the trajectory of system (1) at step k of the process. Let us substitute this function into the functional $V(t, \psi)$. An analog of its derivative due to system (1) is the averaged differences (increments) [1, 2, 9]

$$LV(k, \psi) = E_{k,\psi}[V(k + 1, x_{k+1}(k_0, x_0))] - V(k, \psi),$$

where the operator $E_{k,\psi}$ defines the conditional expectation of the random variable $V(k + 1, x_{k+1}(k_0, x_0))$ given $x_k(k_0, x_0) = \psi$.

In addition, to formulate partial stability conditions, we will also utilize the following auxiliary functionals and functions:

1. Scalar functionals $V^*(k, \psi_y, \psi_{z1})$ and $V^*(\psi_y, \psi_{z1})$, continuous in the domain (3), to indicate an *upper bound* for the functional V and an auxiliary vector function $\mu(k, \psi), \mu(k, 0) \equiv 0$, to correct the construction domain of the functional V. The vector function ψ_{z1} is given by the partition $\psi_z = (\psi_{z1}^{\mathrm{T}}, \psi_{z2}^{\mathrm{T}})^{\mathrm{T}}$ corresponding to the partition $z = (z_1^{\mathrm{T}}, z_2^{\mathrm{T}})^{\mathrm{T}}$. Let us define $||\mu(k, \psi)|| = \sup |\mu(k, \psi(\theta))|$ for $k \in \mathbb{Z}_+,\ \theta \in \mathbb{Z}_0$;

2. A Continuous monotonically increasing in $r > 0$ scalar function $a(r)$, $a(0) = 0$, which specifies standard requirements for the main functional V in the form of a *lower bound*.

The auxiliary function $\mu(k, \psi)$ is introduced since y_1-stability analysis for the "partial" equilibrium $y_k = 0$ of system (1) in the common domain

$$||\psi_{y1}|| < h_1 < h, \quad ||\psi_{y2}|| + ||\psi_z|| < \infty \tag{4}$$

of the function space neither always reveals the desired properties of the functional V nor endows it with these properties. It is rational to study the functional V in the narrower domain

$$||\psi_{y1}|| + ||\mu(k, \psi)|| < h_1 < h, \quad ||\psi_{y2}|| + ||\psi_z|| < \infty \tag{5}$$

based on the following considerations: in fact, the y_1-stability of the "partial" equilibrium $y_k = 0$ of system (1) means satisfaction of the corresponding probability estimates (2) not only for the components of the vector y_1 but also for those of some function $\mu(k, x)$ of the state variables of system (1). In some cases, such a function $\mu(k, x)$ cannot be specified in advance. Therefore, the corresponding function $\mu(k, \psi)$ in the discrete function space should be naturally interpreted as an additional vector function, which is determined when solving the original y_1-stability problem (like a suitable functional V). This leads to the rationale for correcting the construction domain (4) of the functional V via an *additional* auxiliary function $\mu(k, \psi)$.

Theorem 1. *Assume that for system* (1), *it is possible to indicate a functional* V *and an additional vector function* $\mu(k, \psi), \mu(k, 0) \equiv 0$, *so that, for each* $k \in \mathbb{Z}_+$ *and a sufficiently small number* $h_1 > 0$, *the following conditions will hold in the domain* (5) :

$$V(k, \psi) \geqslant a(|\psi_{y1}(0)| + |\mu(k, \psi(0))|), \tag{6}$$

$$V(k, \psi) \leqslant V^*(k, \psi_y, \psi_{z1}), \quad V^*(k, 0, \psi_{z1}) \equiv 0, \tag{7}$$

$$LV(k, \psi) \leqslant 0. \tag{8}$$

Then the "partial" equilibrium $y_k = 0$ of system (1) *is y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20}.*

Theorem 2. *With conditions* (7) *being replaced by*

$$V(k, \psi) \leqslant V^*(\psi_y, \psi_{z1}), \quad V^*(0, \psi_{z1}) \equiv 0, \tag{9}$$

under conditions (6) *and* (8), *the "partial" equilibrium $y_k = 0$ of system* (1) *is uniformly y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20}.*

The proofs of Theorems 1 and 2 are provided in the Appendix.

Remark 3. In Theorems 1 and 2, the auxiliary functional V and its averaged difference (increment) $LV(k, \psi)$ due to system (1) are, generally speaking, *alternating* in the domain (4). Along with the main functional V, the additional auxiliary function μ is introduced for the most rational replacement of the domain (4) with the domain (5). Conditions (7) and (9) identify an *admissible structure* of the functional V, which is determined by the specifics of the partial stability problem posed above: under these conditions, it is possible to use an *arbitrary* continuous functional V^* with $V^*(k, 0, \psi_{z1}) \equiv 0$ or $V^*(0, \psi_{z1}) \equiv 0$ that bounds the functional V from above.

Remark 4. We can take sign-definite (in all variables) *quadratic* functionals (or those of higher order) $V(k, \psi) \equiv V^*(k, \psi_{y1}, \mu(k, \psi))$ as admissible ones. In this case, the choice of the functions μ must be consistent with conditions (7) and (9): for example, the functions of the form $\mu = \mu(\psi_{y2}, \psi_{z1}), \mu(0, \psi_{z1}) \equiv 0$, are admissible.

Remark 5. Let system (1) have the "full" equilibrium $x_k = 0$, $\mu(k, \psi) \equiv 0$, $\xi(k) \equiv 0$, and the initial condition $x_0 \in D_\delta$ be replaced by $||x_0|| < \delta$. Then, under conditions (6) and (8), we arrive at a discrete variant [34, 35] of the classical Rumyantsev theorem [36] on stability with respect to a given part of the variables and its modification for $\mu(k, \psi) \neq 0$ [37]. In this case, conditions (7) and (9) are not required.

Remark 6. Azbelev's scientific school developed another approach to the interpretation and analysis of the stability of stochastic functional-difference systems; for details, see [38].

4. EXAMPLES

We distinguish two classes of nonlinear discrete systems of a given structure, for which partial stability is analyzed in the parameter space. At the same time, we will demonstrate the rationale for using a one-parameter family of functionals.

Example 1. Consider the discrete system (1) composed of the equations

$$\begin{gathered} y_1(k+1) = [a_1 + \alpha_1 \xi_1(k)] y_1(k) + a_2 y_1(k-1) + l y_2(k-1) z_1(k-1) + \alpha_2 y_1(k-1) \xi_2(k), \\ y_2(k+1) = [b + d y_1(k-1)] y_2(k), \\ z_1(k+1) = [c + e y_1(k-1)] z_1(k), \quad z_2(k+1) = Z_2(k, x_k, \xi(k)), \end{gathered} \tag{10}$$

where $\xi_1(k)$ and $\xi_2(k)$ are mutually uncorrelated sequences of independent random variables with identical standard Gaussian distributions for each $k \in \mathbb{Z}_+$; $a_1, a_2, b, c, d, e, l, \alpha_1$, and α_2 are constant parameters. System (10) is a special case of system (1) with $m = 1$; the operator Z_2 satisfies only the general requirements of system (1) for $m = 1$.

System (10) has the "partial" equilibrium

$$y_{1k} = y_{2k} = 0. \tag{11}$$

Consider the family ($M, \beta_1, \beta_2 = \text{const} > 0$) of functionals

$$V(\psi) = \psi_{y1}^2(0) + M \psi_{y2}^2(0) \psi_{z1}^2(0) + (\beta_1 + \alpha_2^2) \psi_{y1}^2(-1) + \beta_2 \psi_{y2}^2(-1) \psi_{z1}^2(-1), \tag{12}$$

representing discrete analogs of the Lyapunov–Krasovskii functionals, and the auxiliary scalar discrete function

$$\mu_1(\psi(\theta)) = \psi_{y2}(\theta)\psi_{z1}(\theta), \quad \theta \in \mathbb{Z}_0 = \{k = -1, 0\}; \tag{13}$$

let $\mu_1(0)$ and $\mu_1(-1)$ denote the values of the function $\mu_1(\psi(\theta))$ for $\theta = 0$ and $\theta = -1$, respectively.

We have the relations

$$\psi_{y1}^2(0) + M\mu_1^2(0) \leqslant V(\psi) = V^*(\psi_{y1}, \psi_{y2}, \psi_{z1}), \quad V^*(0, 0, \psi_{z1}) \equiv 0.$$

The functional V (12) satisfies conditions (6) and (7) in the domain (5), and for all $k \in \mathbb{Z}_+ \cup \mathbb{Z}_0$ its averaged difference (increment) $LV(\psi)$ due to system (10) is defined by

$$\begin{aligned} LV(\psi) &= E_{k,\psi}\{V(\psi(0), X(k, \psi(-1), \psi(0), \xi(k)))\} - V(\psi(-1), \psi(0)) \\ &= E_{k,\psi}\{[a_1\psi_{y1}(0) + a_2\psi_{y1}(-1) + l\psi_{y2}(-1)\psi_{z1}(-1) + \alpha_1\psi_{y1}(0)\xi_1(k) + \alpha_2\psi_{y1}(-1)\xi_2(k)]^2 \\ &\quad + M\psi_{y2}^2(0)\psi_{z1}^2(0)[b + d\psi_{y1}(-1)]^2[c + e\psi_{y1}(-1)]^2\} - \psi_{y1}^2(0) - M\psi_{y2}^2(0)\psi_{z1}^2(0) \\ &\quad +(\beta_1 + \alpha_2^2)[\psi_{y1}^2(0) - \psi_{y1}^2(-1)] + \beta_2[\psi_{y2}^2(0)\psi_{z1}^2(0) - \psi_{y2}^2(-1)\psi_{z1}^2(-1)] \\ &= a_1^2\psi_{y1}^2(0) + 2a_1a_2\psi_{y1}(0)\psi_{y1}(-1) + 2a_1l\psi_{y1}(0)\mu_1(-1) \\ &\quad + 2a_2l\psi_{y1}(-1)\mu_1(-1) + l^2\mu_1^2(-1) + \alpha_1^2\psi_{y1}^2(0) + \alpha_2^2\psi_{y1}^2(-1) + Mb^2c^2\mu_1^2(0) \\ &\quad + r_1\psi_{y1}(-1)\mu_1^2(0) + r_2\psi_{y1}^2(-1)\mu_1^2(0) + r_3\psi_{y1}^3(-1)\mu_1^2(0) + Md^2e^2\psi_{y1}^4(-1)\mu_1^2(0) \\ &\quad - \psi_{y1}^2(0) - M\mu_1^2(0) + (\beta_1 + \alpha_2^2)[\psi_{y1}^2(0) - \psi_{y1}^2(-1)] + \beta_2[\mu_1^2(0) - \mu_1^2(-1)] \\ &= (a_1^2 + \alpha_1^2 + \alpha_2^2 - 1 + \beta_1)\psi_{y1}^2(0) + 2a_1a_2\psi_{y1}(0)\psi_{y1}(-1) + (a_2^2 - \beta_1)\psi_{y1}^2(-1) \\ &\quad + 2a_1l\psi_{y1}(0)\mu_1(-1) + 2a_2l\psi_{y1}(-1)\mu_1(-1) + (Mb^2c^2 - M + \beta_2)\mu_1^2(0) \\ &\quad + (l^2 - \beta_2)\mu_1^2(-1) + r_1\psi_{y1}(-1)\mu_1^2(0) + r_2\psi_{y1}^2(-1)\mu_1^2(0) \\ &\quad + r_3\psi_{y1}^3(-1)\mu_1^2(0) + Md^2e^2\psi_{y1}^4(-1)\mu_1^2(0), \end{aligned}$$

$$r_1 = bcr_0, r_2 = M(b^2e^2 + 4bcde + c^2d^2), r_3 = der_0, r_0 = 2M(be + cd).$$

In these formulas, the conditional expectation has been calculated considering the relations $E[\xi_i(k)] = 0$, $E[\xi_i^2(k)] = 1$, corresponding to the standard Gaussian distributions of the mutually uncorrelated random variables $\xi_i(k)$ $(i = 1, 2)$.

For the sake of simpler analysis, we utilize the inequalities

$$\begin{aligned} 2a_1a_2\psi_{y1}(0)\psi_{y1}(-1) &\leqslant |a_1a_2|[\psi_{y1}^2(0) + \psi_{y1}^2(-1)], \\ 2a_2l\psi_{y1}(-1)\mu_1(-1) &\leqslant |a_2l|[\psi_{y1}^2(-1) + \mu_1^2(-1)] \end{aligned}$$

to obtain the following upper bound for the quadratic part $(LV)_2$ of the above expression for $LV(\psi)$:

$$\begin{aligned} (LV)_2 &\leqslant (a_1^2 + |a_1a_2| + \alpha_1^2 + \alpha_2^2 - 1 + \beta_1)\psi_{y1}^2(0) \\ &\quad + 2a_1l\psi_{y1}(0)\mu_1(-1) + (l^2 + |a_2l| - \beta_2)\mu_1^2(-1) \\ &\quad + (a_2^2 + |a_1a_2| + |a_2l| - \beta_1)\psi_{y1}^2(-1) + (Mb^2c^2 - M + \beta_2)\mu_1^2(0). \end{aligned}$$

Under the conditions

$$\begin{gathered} a_1^2 + |a_1a_2| + \alpha_1^2 + \alpha_2^2 - 1 + \beta_1 < 0, \\ (a_1^2 + |a_1a_2| + \alpha_1^2 + \alpha_2^2 - 1 + \beta_1)(l^2 + |a_2l| - \beta_2) > a_1^2l^2, \\ a_2^2 + |a_1a_2| + |a_2l| - \beta_1 < 0, \quad Mb^2c^2 - M + \beta_2 < 0, \end{gathered}$$

$(LV)_2$ is a negative definite function of the variables $\psi_{y1}(0)$, $\psi_{y1}(-1), \mu_1(0)$, and $\mu_1(-1)$ based on Sylvester's criterion. Therefore, for a sufficiently small number $h_1 > 0$, the averaged difference (increment) $LV(\psi)$ of the functional (12) satisfies the inequality $LV(\psi) \leqslant 0$ in the domain (5).

Assuming that the parameters of system (10) satisfy the conditions

$$\begin{gathered}(|a_1| + |a_2|)^2 + |a_2 l| + a_1^2 + a_2^2 < 1,\\ [(|a_1| + |a_2|)^2 + |a_2 l| + \alpha_1^2 + \alpha_2^2 - 1][l^2 + |a_2 l| + M(b^2c^2 - 1)] > a^2 l^2,\end{gathered} \tag{14}$$

we choose the parameters β_1 and β_2 of the functional (12) as

$$\beta_1 = a_2^2 + |a_1 a_2| + |a_2 l| + \varepsilon_1, \quad \beta_2 = M(1 - b^2c^2) - \varepsilon_2.$$

For sufficiently small numbers $\varepsilon_1,\ \varepsilon_2$, and $h_1 > 0$, we have $LV(\psi) \leqslant 0$ in the domain (5) (*but not in the domain* (4)) for any values of the parameters d and e. Hence, besides conditions (6) and (7), condition (8) holds for the functional V (12) in the domain (5).

Based on Theorem 2, under conditions (14), the "partial" equilibrium (11) of system (10) is uniformly y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20}. The operator $LV(\psi)$ associated with system (10) is *alternating* in the domain (4).

Note that given combinations of the parameters of system (10) can be included in (or excluded from) the stability domain by choosing an appropriate number M. For example, if $l^2 + |a_2 l| = 1$, then the seemingly "natural" choice $M = 1$ in the functional (12) makes the stability domain (14) an empty set for any values of the parameters b, c, d, and e. However, in the same case $l^2 + |a_2 l| = 1$, we may consider the domain (14) by setting $M = 2$.

On the other hand, given a number M, the y_1-stability domain can be changed by modifying the estimates for the quadratic part $(LV)_2$ of the expression defining $LV(\psi)$. Indeed, with the inequality

$$2a_1 l\psi_{y1}(0)\mu_1(-1) \leqslant |a_1 l|[\psi_{y1}^2(0) + \mu_1^2(-1)],$$

in the case of $a_2 = 0$ and a sufficiently small number $h_1 > 0$, the inequality $LV(\psi) \leqslant 0$ will hold in the domain (5) under the conditions

$$\begin{gathered}a_1^2 + \alpha_1^2 + \alpha_2^2 + |a_1 l| - 1 + \beta_1 \leqslant 0,\\ l^2 + |a_1 l| - \beta_2 \leqslant 0, \quad M(b^2c^2 - 1) + \beta_2 \leqslant 0.\end{gathered}$$

In this case, the y_1-stability domain is given by

$$(|a_1| + |l|)^2 + \alpha_1^2 + \alpha_2^2 < 1 + M(1 - b^2c^2), \quad b^2c^2 < 1; \tag{15}$$

in contrast to the domain (14), for $M = l^2 = 1$ the domain (15) is a non-empty set. For illustrative purposes, note that for $M = 1$ and $b^2c^2 = \alpha_1^2 + \alpha_2^2 = 0$, the stability domains (14) and (15) have the form $a_1^2 + l^2 < 1$ and $|a_1| + |l| < \sqrt{2}$, respectively; moreover, the domain (15) covers the case $l^2 = 1$.

In addition, for $\alpha_2 = 0$ and $M = 2$, the domain (12) coincides with that of the uniform y_1-stability in probability of the "partial" equilibrium $y_1(k) = y_2(k) = 0$ of system (10) without aftereffect. Such a system was analyzed in [8] using the Lyapunov function $V(x) = y_1^2 + 2y_2^2 z_1^2$ and the auxiliary function $\mu_1 = y_2 z_1$.

As a numerical experiment, we present the results of calculations using the recurrence relations (12) on the interval $k \in [0, 25]$ for $y_i(-1) = y_i(0) = 0.1$ $(i = 1, 2)$, $z_1(-1) = z_1(0) = 1$ and the parameters $a_1 = 1/2$, $b = 3/2$, $a_2 = 0$, $c = 1/3$, and $d = e = l = 1$.

Table

k	$y_1(k)$	$y_2(k)$	$z_1(k)$	$\xi_1(k)$	$\xi_2(k)$	$y_1(k)$	$y_2(k)$	$z_1(k)$
–1	0.1	0.1	1	–	–	0.1	0.1	1
0	0.1	0.1	1	0	0	0.1	0.1	1
1	0.15	0.16	0.4333	–1	0	0.15	0.16	0.4333
2	0.1750	0.2560	0.1877	1	0	0.1251	0.2560	0.1877
3	0.1568	0.4224	0.2094	1	0	0.1735	0.4224	0.2094
4	0.1264	0.4288	0.0954	0	0	0.1926	0.6864	0.0960
5	0.1517	0.7104	0.0468	–1	0	0.1847	1.1487	0.0487
6	0.1168	1.1554	0.0215	–1	0	0.0967	1.9443	0.0256
7	0.0916	1.9084	0.0104	1	0	0.0720	3.2757	0.0133
8	0.0706	3.0854	0.0060	0	0	0.1098	5.2303	0.0057
9	0.0551	4.9107	0.0025	0	0	0.0985	8.2220	0.0023
10	0.0461	7.7127	0.0004	1	0	0.0791	13.236	0.0011
...	...	...	...	...	...	...	...	...
15	0.0050	65.557	5.5×10^{-6}	–1	0	0.0151	127.93	1.1×10^{-5}
...	...	...	...	...	...	...	...	...
20	0.00039	504.13	2.4×10^{-8}	–1	0	0.0010	1005.5	5.4×10^{-8}
...	...	...	...	...	...	...	...	...
25	1.9×10^{-5}	3832.1	4.7×10^{-10}	0	0	2.9×10^{-5}	7649.1	2.2×10^{-10}

For the "undisturbed" case $\xi_{1,2}(k) \equiv 0$, the calculation results are combined on the left-hand side of the table. Under the random disturbances $\xi_1(k)$ and $\xi_2(k)$ whose intensities α_1 and α_2 satisfy condition (14), the sample trajectories are grouped around the undisturbed trajectory focusing along the Oy_2 axis as $k \to \infty$. To assess the effect of random disturbances on the dynamics of system (10), we also provide the calculation results for $\alpha_1 = 1/3$, $\alpha_2 = 0$, and the same parameter values in the case $\xi_2(k) \equiv 0$ and the admissible realization $\xi_1(k)$ on the interval $k \in [0, 25]$ given by the sequence $\{0, -1, 1, 1, 0, -1, -1, 1, 0, 0, 1, 1, -1, 0, 0, -1, 1, 0, -1, 1, -1, 0, 1, -1, 0\}$ (see the right-hand side of the table).

Example 2. Consider the discrete system (1) composed of the equations

$$\begin{gathered} y_1(k+1) = [a_1 + \alpha_1\xi_1(k) + ly_2(k-1)z_1(k-1)]y_1(k) + \alpha_2 y_1(k-1)\xi_2(k), \\ y_2(k+1) = [b + dy_1(k-1)]y_2(k), \\ z_1(k+1) = [c + ey_1(k-1)]z_1(k), \quad z_2(k+1) = Z_2(k, x_k, \xi(k)), \end{gathered} \tag{16}$$

which represent a structural modification of system (10).

To analyze the y_1-stability in probability of the "partial" equilibrium (11) of system (16), we utilize the discrete functional V (12) for $\beta_1 = 0$ and $\beta_2 = M(1 - b^2c^2) - \varepsilon_2$ and the auxiliary discrete function (13).

The quadratic part $[LV(\psi)]_2$ of the averaged difference (increment) $LV(\psi)$ of this functional due to system (16) has the form

$$[LV(\psi)]_2 = (a_1^2 + \alpha_1^2 + \alpha_2^2 - 1)\psi_{y1}^2(0) + (Mb^2c^2 - M + \beta_2)\mu_1^2(0) + (l^2 - \beta_2)\mu_1^2(-1).$$

Therefore, for a sufficiently small number $h_1 > 0$, the inequality $LV(\psi) \leqslant 0$ will hold in the domain (5) for any values of the parameters d, e, l_1, and l_2 if

$$a_1^2 + \alpha_1^2 + \alpha_2^2 < 1, \quad M(b^2c^2 - 1) + \beta_2 < 0, \quad l^2 - \beta^2 < 0.$$

As a result, under the conditions

$$a_1^2 + \alpha_1^2 + \alpha_2^2 < 1, \quad l^2 + M(b^2c^2 - 1) < 1, \tag{17}$$

the “partial” equilibrium (11) of system (17) is uniformly y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20} based on Theorem 2.

Similar to the analysis of system (10), if $l^2 = 1$, then the “natural” choice $M = 1$ in the functional (12) makes the stability domain (17) an empty set for any values of the parameters b, c, d, and e. However, this set can be expanded as well with a suitably chosen number M, e.g., $M = 2$.

5. CONCLUSIONS

This paper has considered a nonlinear system of stochastic functional-difference equations with finite delay subjected to a discrete “white” noise process. For such a system, problems of stability in probability with respect to a given part of the variables of the “partial” trivial equilibrium have been formulated. The discrete vector function defining the initial state of the system has been supposed deterministic.

Sufficient stability conditions have been established using a discrete-stochastic version of the method of Lyapunov–Krasovskii functionals in an appropriate modification. Along with the main discrete functional V, an additional auxiliary discrete function μ (generally speaking, a vector function) has been considered for correcting the construction domain of the functional V in the discrete function space. The rationale for this approach lies in that the resulting functional V and its averaged difference (increment) due to the system under consideration can be alternating.

APPENDIX

Proof of Theorem 1.

Assume that conditions (6)–(8) are valid for each $k \in \mathbb{Z}_+$ and a sufficiently small number $h_1 > 0$ in the domain (5). Let us take arbitrary numbers ε $(0 < \varepsilon < h_1)$ and k_0 and an initial value x_0 from the domain $D_\varepsilon = \{||y_0|| < \varepsilon, ||z_{10}|| \leqslant L, ||z_{20}|| < \infty\}$. Consider the random process $x(k_0, x_0)$ representing the solution of system (1). We denote by k_ε the “integer” moment when this process first leaves the domain $|y_1| \leqslant \varepsilon : k_\varepsilon = \inf\{k : |y_1(k; k_0, x_0)| > \varepsilon\}$ for $k \geqslant k_0$. Letting $t(k) = \min(k_\varepsilon, k); t(k_0) = k_0$, we have the equalities

$$V\left(t(k), x_{t(k)}(k_0, x_0)\right) - V(k_0, x_0) = \sum_{s=k_0}^{k-1} \Delta V\left(t(s), x_{t(s)}(k_0, x_0)\right);$$

$$\Delta V\left(t(s), x_{t(s)}(k_0, x_0)\right) = \Delta V\left(t(s+1), x_{t(s+1)}(k_0, x_0)\right) - \Delta V\left(t(s), x_{t(s)}(k_0, x_0)\right).$$

Due to these equalities, the sequence $v(k)$ of the random variables $v(k) = V(t(k), x_{t(k)}(k_0, x_0))$ generated by the realizations $x(k, \omega), \xi(k, \omega)$ of the random process $x(k), \xi(k)$ defined by system (1) satisfies the “averaged” relations

$$\begin{aligned} &E\left[V\left(t(k), x_{t(k)}(k_0, x_0)\right) - V(k_0, x_0)\right] \\ &= EV\left(t(k), x_{t(k)}(k_0, x_0)\right) - V(k_0, x_0) = \sum_{s=k_0}^{k-1} E\Delta V\left(t(s), x_{t(s)}(k_0, x_0)\right). \end{aligned} \tag{A.1}$$

By the rule for calculating the repeated expectation, from (A.1) it follows that

$$\begin{aligned} &E\Delta V\left(t(s), x_{t(s)}(k_0, x_0)\right) \\ &= E\left\{E_{t(s), x_{t(s)}(k_0, x_0)}\left[V(t(s+1), x_{t(s+1)}(k_0, x_0))\right]\right\} - V\left(t(s), x_{t(s)}(k_0, x_0)\right) \\ &= E\left[LV\left(t(s), x_{t(s)}(k_0, x_0)\right)\right], \end{aligned}$$

and we arrive at the discrete-functional version of Dynkin's formula:

$$EV\left(t(k), x_{t(k)}(k_0, x_0)\right) - V(k_0, x_0) = \sum_{s=k_0}^{k-1} E\left[LV\left(t(s), x_{t(s)}(k_0, x_0)\right)\right].$$

Consequently, based on condition (8),

$$EV\left(t(k), x_{t(k)}(k_0, x_0)\right) \leqslant V(k_0, x_0) < \infty. \tag{A.2}$$

If $k > k_\varepsilon$ (in this case, $t(k) = k_\varepsilon$), we have $|y_1(t(k); k_0, x_0)| = |y_1(k_\varepsilon; k_0, x_0)| \geqslant \varepsilon$. If $k < k_\varepsilon$ (in this case, $t(k) = k$), the Chebyshev–Markov inequality and the estimate (A.2) yield

$$\begin{aligned} P[|y_1(k; k_0, x_0)| > \varepsilon] &\leqslant a^{-1}(\varepsilon)E[a(|y_1(k; k_0, x_0)|)] \\ &\leqslant a^{-1}(\varepsilon)E[a(|y_1(k; k_0, x_0)| + |\mu(k, x(k; k_0, x_0))|)] \\ &\leqslant a^{-1}(\varepsilon)E[V(k, x_{t(k)}(k_0, x_0))] \\ &= a^{-1}(\varepsilon)E[V(t(k), x_{t(k)}(k_0, x_0))] \leqslant a^{-1}(\varepsilon)V(k_0, x_0). \end{aligned} \tag{A.3}$$

The functional V is continuous for each $k \in \mathbb{Z}_+$, $V(t, 0) \equiv 0$, and conditions (7) hold. Therefore, for all $k_0 \geqslant 0$ and any given number $L > 0$, the limit relation

$$\lim_{||y_0|| \to 0} V(k_0, x_0) = 0 \tag{A.4}$$

is valid for $||z_{10}|| \leqslant L$ uniformly in $||z_{20}|| < \infty$.

Hence, for all $k_0 \geqslant 0$ and any given number $L > 0$, inequalities (A.3) and (A.4) lead to the limit relation

$$\lim_{||y_0|| \to 0} P\left[\sup_{k > k_0} |y_1(k; k_0, x_0)| > \varepsilon\right] = 0,$$

holding for $||z_{10}|| \leqslant L$ uniformly in $||z_{20}|| < \infty$. As a result, for each $k_0 \geqslant 0$, any arbitrarily small numbers $\varepsilon > 0$ and $\gamma > 0$, and any given number $L > 0$, there exists a number $\delta(\varepsilon, \gamma, L, k_0) > 0$ such that inequality (2) will hold for all $k \geqslant k_0$ and $x_0 \in D_\delta$. Thus, the "partial" equilibrium $y_k = 0$ of system (1) is y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20}.

Proof of Theorem 2.

If conditions (9) are satisfied instead of conditions (7), for any given number $L > 0$ the limit relation (A.4) will hold for $||z_{10}|| \leqslant L$ uniformly in $||z_{20}|| < \infty$ and, moreover, in $k_0 \geqslant 0$. As a result, for each $k_0 \geqslant 0$, any arbitrarily small numbers $\varepsilon > 0$ and $\gamma > 0$, and any given number $L > 0$, there exists a number $\delta(\varepsilon, \gamma, L) > 0$ independent of k_0 such that inequality (2) will hold for all $k \geqslant k_0$ and $x_0 \in D_\delta$. Thus, the "partial" equilibrium $y_k = 0$ of system (1) is uniformly y_1-stable in probability for large values of z_{10} and on the whole with respect to z_{20}.

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